Engineering Note

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Entrainment Theory for Axisymmetric, Turbulent, Incompressible Boundary Layers

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Nomenclature

C_f = local skin-friction coefficient
H = δ^*/θ
r = distance from the axis of symmetry to the surface of the body of revolution
u = boundary-layer velocity component in the x direction
U = velocity component at the boundary-layer edge in the x direction
v = boundary-layer velocity component in the y direction
V = velocity component at the boundary-layer edge in the y direction
x = arc length measured along a meridian from the stagnation point
y = coordinate measured normal to the surface of the body of revolution
z = coordinate measured along the axis of the body of revolution from the stagnation point
δ = boundary-layer thickness
δ^* = displacement thickness = $\int_0^\delta \left(1 - \frac{u}{U}\right) dy$
ν = kinematic viscosity
θ = momentum thickness = $\int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$

Introduction

THE entrainment theory of Head[1] has developed into one of the more successful integral methods for predicting the development of turbulent boundary layers. Thompson[2] reviewed the integral methods for two-dimensional, incompressible boundary layers and found that Head's entrainment theory gave the best over-all agreement with experimental data. Since the entrainment theory is based on principles of mass conservation, it can be extended to more general situations with comparative ease. A search of the literature revealed that Head's incompressible entrainment theory has been extended to include effects of mass transfer[1] (suction and blowing), three-dimensionality,[3] heat transfer,[4] surface roughness,[5] and rotation.[6,7] It is the purpose of the present analysis to extend Head's entrainment theory to predicting the development of incompressible, turbulent boundary layers on bodies of revolution. Such shapes are employed in the design of many engineering devices such as aircraft fuselages, marine submersibles and cowlings for propulsion units. Of particular interest to the hydrodynamic designer is the drag on the body and the boundary-layer velocity profile entering an aft-mounted propulsion unit. It will be shown that this information can be readily obtained from the results of the entrainment theory.

Many authors have developed boundary-layer prediction methods for flows of this type[8,9] but few have compared their theoretical results with experimental data obtained on a body of revolution with varying radius. A notable exception is Nelson[9] who developed a calculation procedure based on a law of the wall-law of the wake formulation and compared with the data of Freeman.[10]

Analysis

The continuity equation for steady, incompressible turbulent flow over a body of revolution is

$$(1/r)[\partial(ur)/\partial x] + (\partial v/\partial y) = 0$$

where the flow variables appear as time-averaged quantities. Integrating across the boundary layer gives

$$\frac{1}{rU}\frac{d}{dx}\left(\int_0^\delta urdy\right) = \frac{1}{U}\left(U\frac{d\delta}{dx} - V\right) \qquad (1)$$

where the right side is termed the dimensionless rate of entrainment F. Expanding the left side of Eq. (1) and introducing the displacement thickness gives

$$\frac{d(\delta - \delta^*)}{dx} = F - (\delta - \delta^*)\left(\frac{1}{U}\frac{dU}{dx} + \frac{1}{r}\frac{dr}{dx}\right) \qquad (2)$$

which reduces to Head's[1] two-dimensional entrainment equation for a body of constant radius. This relation plus the momentum integral equation for axisymmetric flow

$$\frac{d\theta}{dx} + \theta\left[\frac{(H+2)}{U}\frac{dU}{dx} + \frac{1}{r}\frac{dr}{dx}\right] = \frac{C_f}{2} \qquad (3)$$

complete the system of differential equations.

Following Head,[1] it is assumed that F is a unique function of $H_{\delta-\delta^*}$, where $H_{\delta-\delta^*} = (\delta - \delta^*)/\theta$, which in turn is uniquely related to the conventional form parameter H. That is, $F(H_{\delta-\delta^*})$ and $H_{\delta-\delta^*} = G(H)$. Head presented empirical curves for F and G, whereas Standen[11] offered the following analytical approximations to these curves:

$$F = 0.0306(H_{\delta-\delta^*} - 3.0)^{-0.653} \qquad (4)$$

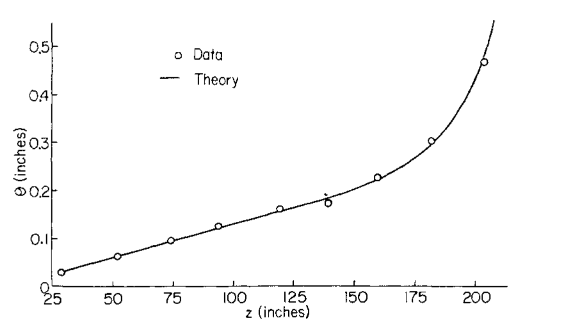

Fig. 1 Momentum thickness prediction compared with data of Freeman.[10]

Received July 20, 1970. The authors express their gratitude to the National Science Foundation for supporting this work through Grant GK-12697.

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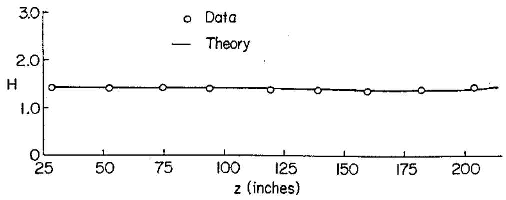

Fig. 2 Shape factor prediction compared with data of Freeman.[10]

and

$$H_{\delta-\delta^*} = 1.535(H - 0.7)^{-2.715} + 3.3 \qquad (5)$$

The system of equations is completed by adopting a skin-friction law such as the Ludwieg-Tillmann relation

$$C_f/2 = 0.123(U\theta/\nu)^{-0.268}\, 10^{-0.678H} \qquad (6)$$

Results and Conclusions

Equations (2) and (3) were numerically integrated, with the aid of the auxiliary relations (4–6), and the results are compared with the data of Freeman[10] in Figs. 1 and 2. Figure 3 compares the measured boundary-layer velocity profile at station 9 with the calculated profile as determined from Thompson's[12] two-parameter family. The results for θ and H are very good but the velocity profile is underpredicted in the middle third of the boundary layer. With an accurate prediction of momentum thickness the total drag on the body of revolution can be determined from Young's[13] method.

Based on these results, it may be concluded that the entrainment theory is a promising approach for predicting the development of turbulent boundary layers on bodies of revolution. Its principal advantages are simplicity and a firm physical basis which enables convenient extension to more complicated situations such as those mentioned in the Introduction.

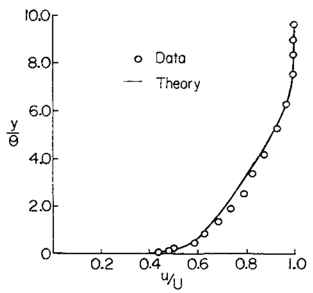

Fig. 3 Boundary-layer velocity profile prediction compared with data of Freeman[10] at station 9.

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